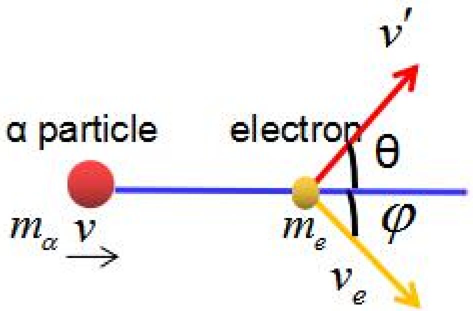
第二章 原子的位形：卢瑟福模型

2.1. A nonrelativistic α particle with velocity v collides with a free electron at rest.Show that the largest angle of deflection of the α particle is approximately $10^{-4} rad$.

速度为v的非相对论的α粒子与一静止的自由电子相碰撞，试证明：α粒子的最大偏离角约为$10^{-4} rad$.

Proof: Suppose the mass of α particle with velocity v is m_α,the velocity of α particle after collision is v',θ is the scattering angle,the mass of an electron is m_e,it rests at origin O before collision,then scatter with velocity v_e,angle φ.By the conservation of energy and momentum in this α-electron system,we have the equations:



$$\frac{1}{2}m_\alpha v^2 = \frac{1}{2}m_\alpha v'^2 + \frac{1}{2}m_e v_e^2 \qquad (1)$$

$$m_\alpha v = m_\alpha v' \cos\theta + m_e v_e \cos\varphi \qquad (2)$$

$$0 = m_\alpha v' \sin\theta - m_e v_e \sin\varphi \qquad (3)$$

$eqn(2)\times\sin\varphi + eqn(3)\times\cos\varphi:$

$$m_\alpha v \sin\varphi = m_\alpha v' \sin(\theta+\varphi)$$

$$\rightarrow m_\alpha v' = m_\alpha v \frac{\sin\varphi}{\sin(\theta+\varphi)} \qquad (4)$$

$eqn(2)\times\sin\theta - eqn(3)\times\cos\theta:$

$$m_\alpha v\sin\theta = m_e v_e \sin(\theta+\varphi)$$

$$\rightarrow m_e v_e = m_\alpha v \frac{\sin\theta}{\sin(\theta+\varphi)} \qquad (5)$$

Substituting eqn(4) and (5) into (1),we obtain:

$$m_\alpha v^2 = m_\alpha v^2 \frac{\sin^2\varphi}{\sin^2(\theta+\varphi)} + \frac{m^2{}_\alpha}{m_e} v^2 \frac{\sin^2\theta}{\sin^2(\theta+\varphi)} \qquad (6)$$

Simplifying eqn(6) becomes:

$$\sin^2(\theta+\varphi) = \sin^2\varphi + \frac{m_\alpha}{m_e}\sin^2\theta \qquad (7)$$

Eqn(7) can be transformed into:

$$\mu\sin^2(\theta+\varphi) = \mu\sin^2\varphi + \sin^2\theta \qquad (8)$$

Where $\mu = \frac{m_e}{m_\alpha}$

θ can be regarded as the function of φ , $\theta(\varphi)$,we obtain the extreme value of eqn(8).

$$2\mu\sin(\theta+\varphi)\cos(\theta+\varphi)d\theta + 2\mu\sin(\theta+\varphi)\cos(\theta+\varphi)d\varphi = 2\mu\sin\varphi\cos\varphi d\varphi + 2\sin\theta\cos\theta d\theta$$

$$\rightarrow \mu\sin 2(\theta+\varphi)d\theta + \mu\sin 2(\theta+\varphi)d\varphi = \mu\sin 2\varphi d\varphi + \sin 2\theta d\theta$$

$$\rightarrow \mu\left[\sin 2(\theta+\varphi) - \sin 2\varphi\right]d\varphi = \left[\sin 2\theta - \mu\sin 2(\theta+\varphi)\right]d\theta$$

$$\rightarrow \frac{d\theta}{d\varphi}\left[\sin 2\theta - \mu\sin 2(\theta+\varphi)\right] = \mu\left[\sin 2(\theta+\varphi) - \sin 2\varphi\right]$$

$$\frac{d\theta}{d\varphi} = 0 \rightarrow \sin 2(\theta+\varphi) - \sin 2\varphi = 0 \rightarrow 2\cos\left[\frac{2(\theta+\varphi)+2\varphi}{2}\right]\sin\left[\frac{2(\theta+\varphi)-2\varphi}{2}\right] = 0$$

$$\rightarrow 2\cos(\theta+2\varphi)\sin\theta = 0$$

If $\sin\theta = 0 \rightarrow \theta = 0$ (minimum) (9)

If $\cos(\theta+2\varphi) = 0 \rightarrow \theta = 90^\circ - 2\varphi$ (10)

Substituting eqn(10) into (8):

$$\mu\sin^2(90^\circ - \varphi) = \mu\sin^2\varphi + \sin^2\theta$$

$\mu\left(\cos^2\varphi-\sin^2\varphi\right)=\sin^2\theta$

$\to\mu\left(\cos^2\varphi-\sin^2\varphi\right)=\sin^2\left(90^\circ-2\varphi\right)=\cos^2\left(2\varphi\right)$

$\to\mu\left(\cos^2\varphi-\sin^2\varphi\right)=\left[\cos^2\varphi-\sin^2\varphi\right]^2$

$\to\mu=\cos^2\varphi-\sin^2\varphi$

$\to\mu=\cos 2\varphi=\cos\left(90^\circ-\theta\right)$

$\to\sin\theta=\mu=\frac{m_e}{m_\alpha}=\frac{m_e}{4m_p}=\frac{1}{4\times1836}=0.000136\approx10^{-4}$

Hence, $\theta\approx10^{-4}rad$ (maximum)

2.2.(a) When α particles with kinetic energy of 5.00MeV are scattered at 90° by gold nuclei,what is the impact parameter? (b) If the thickness of a gold foil is 1.0 μm ,in what percentage of cases will the incident α particles be scattered at angles larger than 90° (that is called back-scattering).

(a)动能为 5.00MeV 的α粒子被金核以90°散射时，它的瞄准距离（碰撞参数）为多大？(b)如果金箔厚为 1.0 μm，则入射α粒子束以大于90°散射（称为背散射）的粒子数是全部入射粒子的百分之几？

Solution:(a) According to the Coulomb scattering formula together with the Coulomb scattering factor:

$$b=\frac{a}{2}\cot\frac{\theta}{2}$$

$$a\equiv\frac{Z_1Z_2e^2}{4\pi\varepsilon_0E}=\left(\frac{e^2}{4\pi\varepsilon_0}\right)\left(\frac{2Z}{E}\right)=1.44\,fm\,MeV\left(\frac{2\times79}{5MeV}\right)=45.5\,fm$$

when $\theta=90^\circ\to\cot\frac{\theta}{2}=1\to b=\frac{a}{2}=22.75\,fm=22.75\times10^{-15}m$

(b) $A_{Au}=197,\rho_{Au}=1.89\times10^{-7}g/m^3$

The probability for α particles hitting a gold foil and scattering into a region $\theta\to\theta-d\theta$ (that is,scattered along a direction lying in $d\Omega$) is:

$$\frac{dN'}{N} = dp(\theta) = \frac{a^2 d\Omega}{16A\sin^4\frac{\theta}{2}} nAt = \frac{a^2 d\Omega}{16A\sin^4\frac{\theta}{2}} nt \qquad (1)$$

Where n is the concentration of nuclei in the foil

$$\rho = m \cdot n = \left(\frac{A}{N_A}\right) n \to n = \frac{\rho N_A}{A}$$

Integrating eqn(1):

$$p(\theta) = \int_{90^\circ}^{180^\circ} \frac{a^2 nt}{16} \cdot \frac{2\pi \sin\theta d\theta}{\sin^4\frac{\theta}{2}} = \frac{\pi}{4} a^2 nt \qquad (2)$$

Substituting the following data,we obtain the probability is:

$$a = 45.5 fm = 45.5 \times 10^{-15} m$$

$$n = \frac{\rho N_A}{A} = \frac{1.89\times10^7 g/m^3 \times 6.02\times10^{23}}{197} = 0.058\times10^{30} g/m^3$$

$$t = 1.0\mu m = 1.0\times10^{-6} m$$

$$\boxed{p(\theta) = 9.4\times10^{-5}}$$

2.4.(a) Suppose the radius of a gold nucleus is 7.0fm;what is the necessary energy for an incident proton just to reach the surface of the nucleus?Generally the proton size is neglected.What is the difference if the proton is taken to have a radius of 1fm?

(b) Suppose we replace the gold nucleus by an aluminum nucleus the radius of which is assumed to be 4.0fm.What is the necessary energy of the incident proton in order that the proton can just reach the surface of the aluminum nucleus in a head-on collision?

(a)假设金核半径为 7.0fm,试问：入射质子需要多少能量，才能在对头碰撞时刚好到达金核的表面？(b)若金核改为铝核，使质子在对头碰撞时刚好到达铝核的表面，那么，入射质子的能量应为多少？设铝核半径为 4.0fm.

Solution:(a)If the proton size is neglected,that is ,the mass of the incident proton m_p is much less than the mass of the gold nucleus

$m_{Au}\left(m_p \le m_{Au}\right)$

Then,the smallest distance r_m between the incident proton and the gold nucleus becomes:

$$r_m = \frac{a}{2}\left(1+\csc\frac{\theta}{2}\right)$$

When $\theta = 180^\circ \to r_m = \frac{a}{2}\left(1+\csc 90^\circ\right) = a$

Then we have: $r_m = \frac{Z_1Z_2e^2}{4\pi\varepsilon_0 E} \to E = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_1Z_2}{r_m} = 1.44 fmMeV \times \frac{1\times 79}{7.0\,fm} = 16.25MeV$

If the proton is taken to have a radius of 1fm,

$$E = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_1Z_2}{r_m + r_p} = 1.44\,fmMeV \times \frac{1\times 79}{(7.0+1.0)\,fm} = 14.22MeV$$

(b)If we replace the gold nucleus by an aluminum nucleus,the energies in the center-of-mass system and the laboratory system are related by:

$$E_c = \frac{1}{1+\gamma}E_L, \gamma = \frac{m_p}{m_{Al}} \to E_c = \frac{m_{Al}}{m_{Al}+m_p}E_l$$

Then ,the smallest distance r_m between the incident proton and the aluminum nucleus becomes:

$$r_m = a_c \equiv \frac{Z_1Z_2e^2}{4\pi\varepsilon_0 E_c} = \frac{Z_1Z_2e^2}{4\pi\varepsilon_0 E_L}\cdot\frac{m_{Al}+m_p}{m_{Al}}$$

$$\to E_L = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_1Z_2}{r_m}\cdot\frac{m_{Al}+m_p}{m_{Al}} = 1.44\,fmMeV \times \frac{1\times 13}{4.0\,fm}\times\frac{27+1}{27} = 4.85MeV$$

如果入射粒子质量并不是远小于靶核质量，那要考虑质心系能量：（质心系中相互作用的两粒子的动能之和）

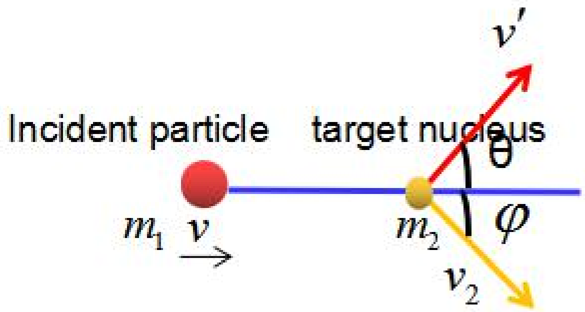
2.8 (a)Incident particles of mass m_1 are elastically scattered by a target nucleus at rest.The mass of the target is m_2 (where $m_2 \le m_1$).Show that the largest possible scattering angle of the incident particle in the laboratory system $\theta_{L,\max}$ is determined by $\sin\theta_{L,\max} = m_2 / m_1$.

(b)If an α particle is scattered by a deuterium nucleus which is originally at rest,what is the maximum angle of scattering in the laboratory system?

(a)质量为m_1的入射粒子被质量为m_2的静止靶核弹性散射，试证明：入射粒子在实验室坐标系中的最大可能偏转角θ_L由下式决定：$\sin\theta_L = m_2/m_1$。

(b)假如α粒子在原来静止的氢核上散射，试问：它在实验室坐标系中最大的散射角为多大？

Proof:(a)By the conservation of energy and momentum in this system,we have the equations:



$$\frac{1}{2}m_1v^2 = \frac{1}{2}m_1v'^2 + \frac{1}{2}m_2v_2{}^2 \quad (1)$$

$$m_1v = m_1v'\cos\theta + m_2v_2\cos\varphi \quad (2)$$

$$0 = m_1v'\sin\theta - m_2v_2\sin\varphi \quad (3)$$

$eqn(2)\times\sin\varphi + eqn(3)\times\cos\varphi$:

$$m_1v' = m_1v\frac{\sin\varphi}{\sin(\theta+\varphi)} \quad (4)$$

$eqn(2)\times\sin\theta - eqn(3)\times\cos\theta$:

$$m_2v_2 = m_1v\frac{\sin\theta}{\sin(\theta+\varphi)} \quad (5)$$

Substituting eqn(4) and (5) into (1),we obtain:

$$m_1v^2 = m_1v^2 \frac{\sin^2\varphi}{\sin^2(\theta+\varphi)} + \frac{m^2{}_1}{m_2}v^2 \frac{\sin^2\theta}{\sin^2(\theta+\varphi)} \qquad (6)$$

Simplifying eqn(6) becomes:

$$\sin^2(\theta+\varphi) = \sin^2\varphi + \frac{m_1}{m_2}\sin^2\theta \qquad (7)$$

Eqn(7) can be transformed into:

$$\mu\sin^2(\theta+\varphi) = \mu\sin^2\varphi + \sin^2\theta \qquad (8)$$

Where $\mu = \frac{m_2}{m_1}$

θ can be regarded as the function of φ , $\theta(\varphi)$,we obtain the extreme value of eqn(8).

$$\frac{d\theta}{d\varphi}\left[\sin 2\theta - \mu\sin 2(\theta+\varphi)\right] = \mu\left[\sin 2(\theta+\varphi) - \sin 2\varphi\right]$$

$$\frac{d\theta}{d\varphi} = 0 \to \sin 2(\theta+\varphi) - \sin 2\varphi = 0 \to 2\cos(\theta+2\varphi)\sin\theta = 0$$

If $\sin\theta = 0 \to \theta = 0$ (minimum) (9)

If $\cos(\theta+2\varphi) = 0 \to \theta = 90^\circ - 2\varphi$ (10)

Substituting eqn(10) into (8):

$$\mu\sin^2(90^\circ - \varphi) = \mu\sin^2\varphi + \sin^2\theta$$

$$\mu(\cos^2\varphi - \sin^2\varphi) = \sin^2\theta$$

$$\to \mu(\cos^2\varphi - \sin^2\varphi) = \sin^2(90^\circ - 2\varphi) = \cos^2(2\varphi)$$

$$\to \mu(\cos^2\varphi - \sin^2\varphi) = \left[\cos^2\varphi - \sin^2\varphi\right]^2$$

$$\to \mu = \cos^2\varphi - \sin^2\varphi$$

$$\to \mu = \cos 2\varphi = \cos(90^\circ - \theta)$$

$$\to \sin\theta_L = \mu = \frac{m_2}{m_1}$$

(b) The mass of the α particle is double size of the deuterium

nucleus,that is,氦核相对原子质量是氘核的 2 倍

$$\sin\theta_L=\mu=\frac{m_D}{m_\alpha}=\frac{1}{2}\rightarrow\theta_L=30^\circ$$

2.10.An accelerator-produced proton beam with an energy of 1.2MeV and a beam-current of 5.0 nA ,impinges perpendicularly upon a gold foil of thickness 1.5 μm .Find the number of protons that are scattered in 5 min by the foil into the following intervals of angles:

(a) $59-61^\circ$

(b) $\theta>\theta_0=60^\circ$

(c) $\theta<\theta_0=10^\circ$

由加速器产生的能量为 1.2MeV、束流为 5.0nA 的质子束，垂直地射到厚为 1.5 μm 的金箔上，试求 5min 内被金箔散射到下列角间隔内的质子数：

Solution :(a)The total incident particles number in 5 min is:

$$N=\frac{Q}{e}=\frac{It}{e}=\frac{5.0\times10^{-9}A\times5\times60s}{1.602\times10^{-19}C}=9.36\times10^{12}$$

Then the number of scattered particles measured in the direction of $d\Omega$ should be :

$$dN'=Ndp(\theta)=N\frac{a^2d\Omega}{16A\sin^4\frac{\theta}{2}}nAt=ntN\left(\frac{a}{4}\right)^2\frac{d\Omega}{\sin^4\frac{\theta}{2}} \qquad (1)$$

Where n is the concentration of nuclei in the foil:

$$\rho=m\cdot n=\left(\frac{A}{N_A}\right)n\rightarrow n=\frac{\rho N_A}{A}$$

Integrating eqn(1) :

$$\int_{\theta_1}^{\theta_2} dN' = \int_{\theta_1}^{\theta_2} ntN\left(\frac{a}{4}\right)^2 \frac{d\Omega}{\sin^4\frac{\theta}{2}} = ntN\left(\frac{a}{4}\right)^2 \int_{\theta_1}^{\theta_2} \frac{2\pi\sin\theta d\theta}{\sin^4\frac{\theta}{2}}$$

$$= ntN\left(\frac{a}{4}\right)^2 (4\pi)\int_{\theta_1}^{\theta_2} \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} d\theta = ntN\left(\frac{a}{4}\right)^2 (4\pi)\int_{\theta_1}^{\theta_2} \frac{2d\sin\frac{\theta}{2}}{\sin^3\frac{\theta}{2}}$$

$$= ntN\left(\frac{a}{4}\right)^2 (-4\pi)\left[\frac{1}{\sin^2\theta}\right]_{\theta_1}^{\theta_2}$$

Substituting the following data,we obtain the number of proton that are scattered in 5min by the foil in this interval of angles :

$$N = 9.36\times10^{12}$$

$$n = \frac{\rho N_A}{A} = \frac{1.89\times10^7 g/m^3\times6.02\times10^{23}}{197} = 0.058\times10^{30} g/m^3$$

$$t = 1.5\mu m = 1.5\times10^{-6} m$$

$$a \equiv \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)\left(\frac{1\times79}{E}\right) = 1.44 fmMeV\times\left(\frac{79}{1.2MeV}\right) = 94.8fm = 94.8\times10^{-15}m$$

$$\left[\frac{1}{\sin^2\frac{\theta}{2}}\right]_{59^\circ}^{61^\circ} = -0.242$$

$$N' = 1.39\times10^9$$

(b)the number of proton that are scattered in 5min by the foil in this interval of angles :

$$\left[\frac{1}{\sin^2\frac{\theta}{2}}\right]_{60^\circ}^{180^\circ} = -3$$

$$N' = 1.72\times10^{10}$$

(c)the number of proton that are scattered in 5min by the foil in this interval of angles :

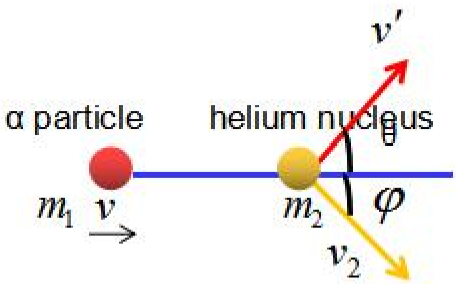
$$\left[\frac{1}{\sin^2\frac{\theta}{2}}\right]_{10^\circ}^{180^\circ} = -131$$

$$N' = 7.5\times10^{11}$$

$$N'' = N - N' = 9.36\times10^{12} - 7.5\times10^{11} = 8.6\times10^{12}$$

2.18 When an α particle is scattered by a helium nucleus at rest,what is the largest scattering angle in the laboratory system?

Solution:(a)By the conservation of energy and momentum in this system,we have the equations:



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$$m_1v = m_1v'\cos\theta + m_2v_2\cos\varphi \quad (2)$$

$$0 = m_1v'\sin\theta - m_2v_2\sin\varphi \quad (3)$$

$eqn(2)\times\sin\varphi + eqn(3)\times\cos\varphi$:

$$m_1v' = m_1v\frac{\sin\varphi}{\sin(\theta+\varphi)} \quad (4)$$

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$$m_2v_2 = m_1v\frac{\sin\theta}{\sin(\theta+\varphi)} \quad (5)$$

Substituting eqn(4) and (5) into (1),we obtain:

$$m_1 v^2 = m_1 v^2 \frac{\sin^2 \varphi}{\sin^2(\theta+\varphi)} + \frac{m^2{}_1}{m_2} v^2 \frac{\sin^2 \theta}{\sin^2(\theta+\varphi)} \qquad (6)$$

Simplifying eqn(6) becomes:

$$\sin^2(\theta+\varphi) = \sin^2 \varphi + \frac{m_1}{m_2}\sin^2 \theta \qquad (7)$$

Eqn(7) can be transformed into:

$$\mu \sin^2(\theta+\varphi) = \mu \sin^2 \varphi + \sin^2 \theta \qquad (8)$$

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$$\frac{d\theta}{d\varphi}[\sin 2\theta - \mu \sin 2(\theta+\varphi)] = \mu[\sin 2(\theta+\varphi) - \sin 2\varphi]$$

$$\frac{d\theta}{d\varphi} = 0 \rightarrow \sin 2(\theta+\varphi) - \sin 2\varphi = 0 \rightarrow 2\cos(\theta+2\varphi)\sin\theta = 0$$

If $\sin\theta = 0 \rightarrow \theta = 0$ (minimum) (9)

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$$\mu(\cos^2 \varphi - \sin^2 \varphi) = \sin^2 \theta$$

$$\rightarrow \mu(\cos^2 \varphi - \sin^2 \varphi) = \sin^2(90^\circ - 2\varphi) = \cos^2(2\varphi)$$

$$\rightarrow \mu(\cos^2 \varphi - \sin^2 \varphi) = [\cos^2 \varphi - \sin^2 \varphi]^2$$

$$\rightarrow \mu = \cos^2 \varphi - \sin^2 \varphi$$

$$\rightarrow \mu = \cos 2\varphi = \cos(90^\circ - \theta)$$

$$\rightarrow \sin \theta_L = \mu = \frac{m_2}{m_1}$$

The mass of the α particle equals to the mass of a helium nucleus

$$\sin\theta_L = \mu = \frac{m_2}{m_1} = 1 \rightarrow \theta_L = 90^\circ$$